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AN OPTIMAL ALGORITHM FOR CONSTRUCTING THE WEIGHTED VORONOI DIAGRAM IN THE PLANE

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Abstract—Let S denote a set of n points in the plane such that each point p has assigned a positive weight $w(p)$ which expresses its capability to influence its neighbourhood. In this sense, the weighted distance of an arbitrary point x from p is given by $d_e(x, p)/w(p)$ where d_e denotes the Euclidean distance function. The weighted Voronoi diagram for S is a subdivision of the plane such that each point p in S is associated with a region consisting of all points x in the plane for which p is a weighted nearest point of S.

An algorithm which constructs the weighted Voronoi diagram for S in $O(n^2)$ time is outlined in this paper. The method is optimal as the diagram can consist of $\Theta(n^2)$ faces, edges and vertices.

Voronoi diagram Weighted points Geometric transform Cell complex Incremental construction Concrete complexity

1. INTRODUCTION

Let S denote a finite set of points in the Euclidean d-dimensional space. The Voronoi diagram of S is a well-known structure which makes explicit some proximity information about S. The broad scope of applications of the diagram is best documented by the various rediscoveries of the Voronoi diagram in different areas of science. As far as we know Voronoi[1] was the first to look at the diagram for examining quadratic forms. Later, the diagram was used for applications in physics,[2] in geography,[3] in vision and biology,[4,5] in archeology[6] and other areas. Shamos[7] and Shamos and Hoey[8] introduced the two-dimensional Voronoi diagram to computational geometry and demonstrated an efficient algorithm for constructing it. Efficient algorithms for three- and higher-dimensional point-sets follow from the results in Seidel[9] and the close relationship between Voronoi diagrams in d dimensions and convex hulls in $d + 1$ dimensions (see Brown[10]).

Also, generalizations of the diagram were considered by several authors. Shamos and Hoey[8] introduced Voronoi diagrams of higher order and Drysdale and Lee,[11] as well as Kirkpatrick,[12] looked at diagrams for more general objects than points. This paper concentrates on a different generalization by assigning to each point p of the given set a positive *weight* $w(p)$ which expresses the power of p to influence its neighbourhood. Original Voronoi diagrams are the special case where all points are equally powerful. The new structure is called the weighted Voronoi diagram of a set of weighted points. This concept of weighting the points has already been considered in Boots,[13] who concentrated on applications in geography. Our major interest is in the two-dimensional case. Thus, let S denote a finite set of points in the Euclidean plane. The *weighted distance* $d_w(x, p)$ between an arbitrary point x in the plane and a point p in S equals $d_e(x, p)/w(p)$, d_e denoting the Euclidean distance function. The *weighted Voronoi diagram of* S (for short *WVD*(S)) is a subdivision of the plane consisting of faces, edges and vertices. Let *region*(p) denote the *region* (*of influence*) *of* a point p in S, i.e.

$$region(p) = \{x \mid d_w(x, p) \le d_w(x, q),\ q \text{ in } S\}.$$

A *face* of the WVD(S) is a connected component of the interior of a thus defined region. An *edge* is the relative interior of the intersection of two closed faces, and a *vertex* is an endpoint of an edge (see Fig. 1 for an illustration). Clearly, if the regions of two points p and q have a non-empty intersection, then this intersection is a subset of the curve defined by the *WVD-equation*

$$d_w(x, p) = d_w(x, q)$$

for p and q.

The practical relevance of the planar WVD was reported in Boots,[13] who refers to Gambini *et al.*,[14] Huff and Jenks[15] and Huff[16] for the majority of applications of the WVD. To aid the intuition of the reader, let us describe briefly two practical situations where the planar WVD comes in.

In some geographic area of interest we consider a collection of concurrent shops. Each shop has assigned to it some number to express its power of attracting customers. Big shops with a large variety of articles usually attract customers with more power than small shops. Nevertheless, a small shop will realize the dominating attraction in its near neighbourhood. The WVD for the shops seems to capture this information in an appropriate way and to make explicit the area of dominating attraction for each shop.

Another problem leading to the same model involves a set of transmitters with varying strength. It is of interest to determine for some arbitrary given point x the transmitter which is received best in x or to construct the region of points in which a transmitter p is received best among some finite collection. Underlying these questions is the physical assumption that a transmitter p with strength $w(p)$ is received with strength $w(p)/d_e(x,p)^2$ at a point x. Although the influence of a transmitter thus decreases with the square of the distance, again the WVD for the transmitters captures the weighted proximity information. This was observed by Schoone and van Leeuwen,[17] who examined the case of only two kinds of transmitters, the transmitters of one kind having the same strength.

The organization of this paper is as follows. Section 2 discusses some properties of the two-dimensional WVD relevant for constructing it. Section 3 introduces a geometric transform which embeds the two-dimensional WVD in three dimensions. The algorithm for constructing the WVD is outlined in Section 4. Finally, Section 5 reviews the main contributions of the paper and discusses several extensions of our results.

2. SOME PROPERTIES OF THE TWO-DIMENSIONAL WVD

The primary concern of this section is the demonstration of some basic properties of the WVD in the plane.

Observation 2-1. Let $S = \{p, q\}$ consist of two weighted points in the plane and let $w(p) < w(q)$. Then the region of influence of p is the closed disc with center

$$(w^2(p)\,p - w^2(q)\,q)/(w^2(p) - w^2(q))$$

and radius

$$(w(p)\,w(q)d_e(p,q))/(w^2(p) - w^2(q)).$$

The region of influence of q is the closed complement of this disc.

The above formulae are derived from the WVD-equation for p and q. Observe that the ratio $w(p)/w(q)$ determines the disc rather than the individual weights of p and q themselves. For convenience, we call the circle which is the intersection of both regions of influence the *separation of p and q*, for short $sep(p,q)$. The closed interior of $sep(p,q)$ is termed the *dominance of p over q*, for short $dom(p,q)$, and the closed complement of $dom(p,q)$ is termed $dom(q,p)$. These definitions are easily extended to the case $w(p) = w(q)$, where the separating circle degenerates to a line. Without loss of generality, we will draw no distinction between circular or spherical objects and their affine degeneracies.

Observation 2-2. Let S denote a finite set of weighted points in the plane and let p be in S. Then

$$region(p) = \bigcap_{q \in S - \{p\}} dom(p,q).$$

It is clear that the regions of a set S of weighted points cover the whole plane, since for each point x of the plane there is at least one weighted nearest point in S. The intersection of two regions of influence is a circle or part of a circle. Thus, the WVD(S) is a subdivision of the plane with circular edges. If a point x falls into a face of the subdivision then there is a unique weighted nearest point. If x is on an edge then there are two weighted nearest points. Finally, if x is a vertex then there are at least three. Figure 1 shows the WVD of a set of 8 points and contains 9 faces, 16 edges and 9 vertices. The weights of the points are in parentheses.

The above discussion, as well as easy analytic calculations, imply observation 2-3.

Observation 2-3. Let p, q and r denote three weighted points in the plane. Then there are at most two points common to $sep(p,q)$, $sep(q,r)$ and $sep(p,r)$ and a point common to two of them is common to all three.

The example depicted in Fig. 1 reveals some unpleasant properties of the diagram: the region of a

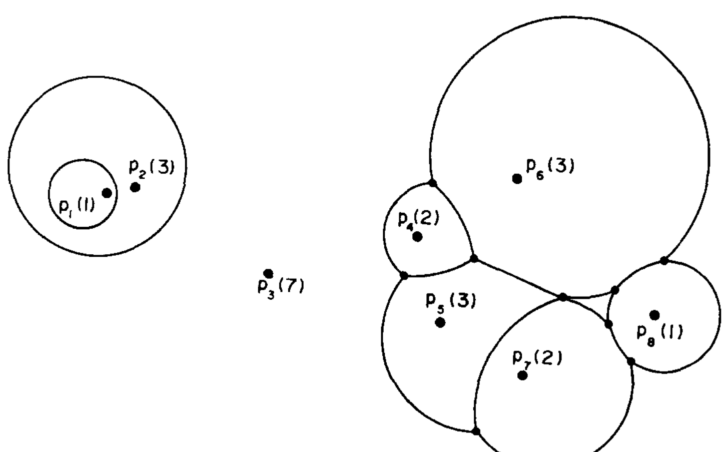

Fig. 1. The WVD for 8 points.

point need neither be connected nor need its connected parts be simply connected (see, for example, the region of p_3). Needless to say, the faces need not be convex nor need the collection of edges form a connected component.

Lemma 2–4. Let S denote a set of n weighted points in the plane. Then the WVD(S) contains $\Omega(n^2)$ faces, edges and vertices in the worst case.

Proof. We show the assertion by exhibiting an example which is realizable for all positive integers n. $\lfloor n/2 \rfloor$ of the points in S are chosen collinear and with identical weights. These points induce a diagram consisting of $\lfloor n/2 \rfloor - 1$ parallel lines. The remaining $\lceil n/2 \rceil$ points are chosen such that: (1) no two regions of them share an edge; (2) their regions are convex; (3) each of these regions shares an edge with each region of the former $\lceil n/2 \rceil$ points (see Fig. 2). It is readily seen that it is possible to choose the latter points such that (1), (2) and (3) hold and there are $\Theta(n^2)$ faces, edges and vertices in the resulting diagram, which completes the argument.

An obvious question is whether or not the given bound is asymptotically tight. This question is settled in the affirmative by the following assertion. The proof is given in Section 3.

Lemma 2–5. Let S be a set of n weighted points in the plane. Then the region of a point p in S is bounded by $O(n)$ edges.

As an immediate consequence of Lemma 2–5 the WVD(S) contains at most $O(n^2)$ edges and thus $O(n^2)$ faces and vertices.

3. EMBEDDING THE WVD IN THREE DIMENSIONS

A geometric transform is described in this section which leaves us with a three-dimensional problem, in some aspects preferable to constructing the planar WVD directly. Intuitively, each weighted point is associated with a point and a convex polyhedron in three dimensions, such that the region of the original point can be obtained by transforming the intersection of the polyhedron with a sphere. The geometric transform employed was first used by Brown[10] for solving a variety of other geometric tasks.

The plane which contains the given set S of weighted points is identified with the XY-plane in three dimensions. Next, a point I not in the XY-plane is distinguished. Let p and q denote two points in S and recall that p and q define a circle $sep(p, q)$. There exists a unique sphere, called the *sphere of p and q* (for short $sph(p, q)$), whose intersection with the XY-plane is exactly $sep(p, q)$ and which contains I (see Fig. 3). Without loss of generality, let $w(p) < w(q)$ such that p is inside of $sph(p, q)$. Then the closed interior of $sph(p, q)$ is called the *spherical domination of p over q* (for short $sdom(p, q)$). Similarly, the closed complement of $sdom(p, q)$ is termed $sdom(q, p)$. Now we define the *spherical region of p* (for short $sregion(p)$) as the intersection of all $sdom(p, q)$, for q in S different from p.

Trivially, two spherical regions intersect at most in a two-dimensional variety. Now the question arises whether or not the spherical regions define a partition of the whole space. This is settled in the affirmative as it

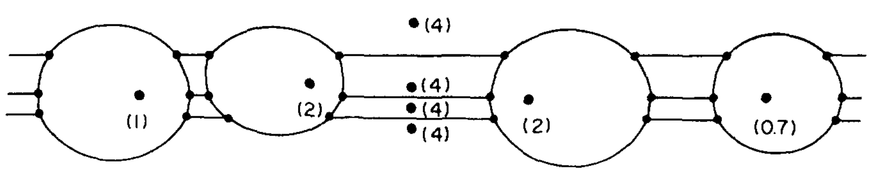

Fig. 2. Worst-case configuration for $n = 8$.

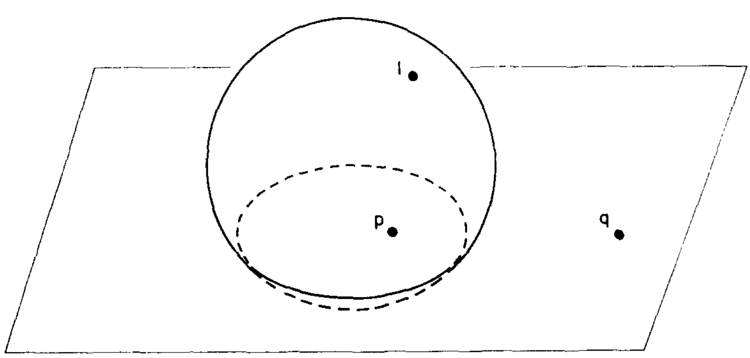

Fig. 3. The spherical domination.

is an immediate consequence of the following lemma. We omit the proof since it can be done by straightforward calculations.

Lemma 3–1. Let p, q and r denote three weighted points in the XY-plane and let x be an arbitrary point in the intersection of $sdom(p, q)$ and $sdom(q, r)$. Then x is also in $sdom(p, r)$.

Now the geometric transform often called *inversion* is applied to the spherical partition of the three-dimensional space. We choose I for the *center of* the *inversion* and map each point x whose distance from I is $d_e(x, I)$ into the point x' collinear to x and I such that $d_e(x', I) = 1/d_e(x, I)$. Simple geometric arguments show that a plane is thus mapped into a sphere containing I and vice versa (see, for example, Brown[10]). Note that the spheres defined for the points in S are thus mapped into planes and the spherical dominations are mapped into closed halfspaces. Moreover, the spherical regions of the points in S are mapped into polyhedra which are convex since they come from intersecting halfspaces. Lemma 3–1, which implies that the spherical regions define a partition of the space, thus implies that the polyhedra do the same. We call the polyhedron which is obtained by inverting $sregion(p)$ of some point p in S as the *polyhedron of* p, (for short $poly(p)$). Trivially, the inverted point p' of p is contained in the interior of $poly(p)$.

Recall that the WVD(S) is the intersection of the partition defined by the spheres of the points and the XY-plane. Thus, the WVD(S) is mapped by inversion into the intersection of the partition defined by the polyhedra of the points in S with the sphere corresponding to the XY-plane. As the inversion is involutory, the WVD(S) can be obtained by re-inverting the latter intersection. With these correspondences in mind it is easy to prove Lemma 2–5 of the preceding section.

Proof of Lemma 2–5. The region of a weighted point p in the WVD(S) is the inverted picture of the intersection of $poly(p)$ with the sphere corresponding to the XY-plane. By definition of $region(p)$, $poly(p)$ is the intersection of n-1 halfspaces. Thus $poly(p)$ has $O(n)$ faces, edges and vertices and the intersection of $poly(p)$ with a sphere consists of $O(n)$ faces, edges and vertices on this sphere. The assertion follows from the fact that inverting does not change the number of faces, edges and vertices involved. This completes the argument.

The presented correspondences reveal the idea of our algorithm for computing the WVD(S), which is constructing the *cell complex* consisting of the polyhedra of the points in S, intersecting the cell complex with the sphere corresponding to the XY-plane and finally inverting this intersection. Figure 4 gives an illustration of the correspondence between the two intersections. It displays the region of a weighted point in the XY-plane, as well as the intersection of its polyhedron and the sphere corresponding to the XY-plane.

4. CONSTRUCTING THE WVD

In this section an algorithm is outlined which constructs the WVD of a given finite set of weighted points in the plane. The major part of the algorithm constructs the cell complex $C(S)$ which consists of the polyhedra of the weighted points. We discuss the construction of $C(S)$ first and come back to the WVD later.

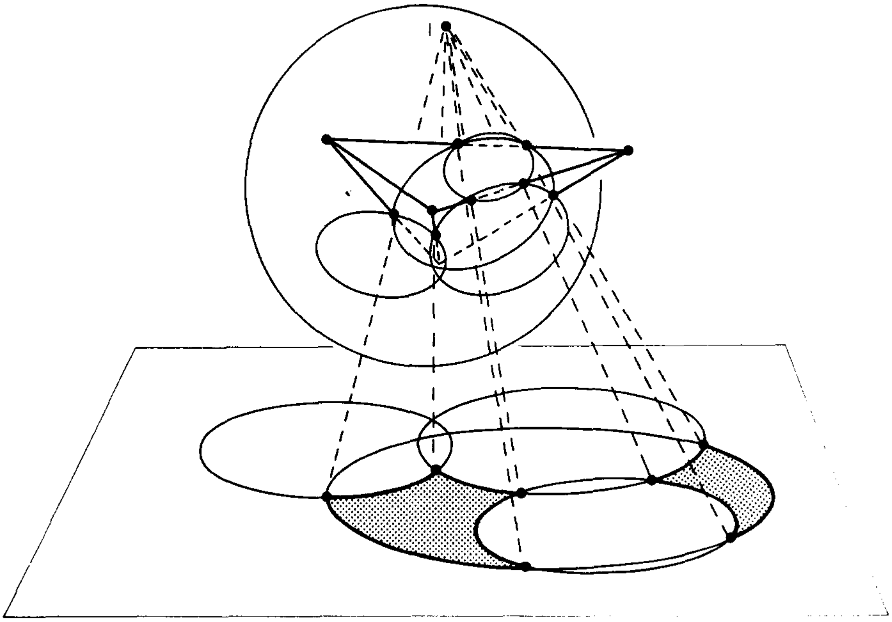

Fig. 4. The three-dimensional embedding.

The *cells of* $C(S)$ are the interiors of the polyhedra of the weighted points in S. A *face of* $C(S)$ is the relative interior of the intersection of two closed cells, an *edge of* $C(S)$ is the relative interior of the intersection of two closed faces and a *vertex of* $C(S)$ is the intersection of two closed edges. A cell and a face (or a face and an edge or an edge and a vertex) are called *incident* if the closure of the cell (or face or edge) contains the face (or edge or vertex).

Out of a number of possibilities to store $C(S)$ we choose the *incidence lattice of* $C(S)$, which consists of a name for each cell, face, edge and vertex of $C(S)$ and connects incident objects (see Gruenbaum[18] for cell complexes in general and the incidence lattice in particular). Note that this representation, which is also used in Seidel,[9] reflects little of the ordering inherent in $C(S)$. To remedy this serious shortcoming we store the edges incident with a face in the natural ordering around the face. Similarly, the faces incident with an edge are stored in order. In addition, pointers are established such that given a face f and an incident edge e, the adjacent two edges incident with f and the adjacent two faces incident with e are available in constant time. It is worthwhile to mention that storing the coordinates of the vertices or the positions of the planes determining the faces suffices to fix the cell complex in space.

The incidence lattice allows for an efficient algorithm to intersect a cell c in the complex with a plane pl. This is demonstrated in the following, where it is assumed that an edge e_0 of c and a face f_0 of c, incident with e_0, which intersect pl are known. For simplicity, we also assume that pl contains no face, edge or vertex of c. These cases can be incorporated without affecting the asymptotic runtime. The algorithm constructs the cyclic sequence of edges and vertices defining the intersection of c and pl. With each edge or vertex x, the face or edge y of c is associated such that x is the intersection of y and pl. These details are omitted in the description of the algorithm.

Algorithm CELL-PLANE

Let $e := e_0$ be the current edge and let $f := f_0$ denote a face of c incident with e.

Step 1. Scan the edges around f which are below pl, say, until an edge e' different from e is encountered (if it exists) which intersects pl.

Case 1.1. e' exists. Then the process is finished if $e' = e_0$. Otherwise, let f' be the face of c different from f and incident with e. Set $e := e'$ and $f := f'$ and repeat Step 1.

Case 1.2. e' does not exist, which is only possible if f is unbounded. If this case occurs the second time then the process is finished. Otherwise set $e := e_0$ and let the face f of c be different from f_0 and incident with e. Repeat Step 1.

Lemma 4–1. Let c be a cell and let pl be a plane. If an edge e_0 of c and an incident face f_0 of c intersecting pl are given, then Algorithm CELL-PLANE intersects c and pl in time $O(n + h)$, where n denotes the number of edges of c intersecting pl and h denotes the number of edges of c on some fixed side of pl.

We omit the proof since the algorithm is extremely simple. Care is only required for verifying that the presentation of c as a cell in a cell complex allows the scanning in constant time per step. Using Algorithm CELL-PLANE we construct the cell complex $C(S)$ for a set S of n weighted points in the XY-plane. To this end, we call the inverted picture of $sph(p, q)$ the *plane of p and q* (for short $pl(p, q)$). Clearly, $pl(p, q)$ can be computed in constant time from p and q. The algorithm to be presented constructs $C(S)$ by successively inserting the weighted points.

Algorithm CELL COMPLEX

Let $S = \{p_1, \ldots, p_n\}$ and let $S_i = \{p_1, \ldots, p_i\}$, for $i = 1, \ldots, n$. $C(S_i)$ is constructed from $C(S_{i-1})$ by intersecting the cells in $C(S_{i-1})$ with the planes $pl(p_i, p_j)$, $j = 1, \ldots, i-1$. Let p^* denote a point in S_{i-1} which is nearest to p_i, i.e. p_i is in $poly(p^*)$ in $C(S_{i-1})$. An edge e_0 and an incident face f_0 of the cell of $poly(p^*)$ which intersect $pl(p_i, p^*)$ are determined. The cell, together with e_0 and f_0, are put into an initially empty queue Q and the construction of $poly(p_i)$ proceeds as follows.

While Q is not empty the first cell c with its edge e and face f are taken from Q. Let p denote the point in S_{i-1} such that c is the interior of $poly(p)$ in $C(S_{i-1})$. Now Algorithm CELL-PLANE is used to intersect c with $pl(p_i, p)$. The side of $pl(p_i, p)$ defined by p_i is chosen for the side on which the edge-sequences of c are scanned. When a face f' of c is intersected with $pl(p_i, p)$ then the cell c' different from c which is incident with f' is put into Q, unless c' was already put into Q at the time $poly(p_i)$ was constructed. If c' is put into Q then c' is put together with f' and an edge e' incident with f' which intersects $pl(p_i, p)$.

At last, when $poly(p_i)$ is completed in $C(S_i)$ then the faces, edges and vertices in the interior of $poly(p_i)$ are deleted from the constructed cell complex. This finally gives $C(S_i)$.

Let us analyze the amount of time required by the algorithm to construct the cell complex $C(S)$.

Lemma 4–2. Let S be a set of n weighted points in the XY-plane. Then Algorithm CELL COMPLEX constructs $C(S)$ in $O(n^2)$ time.

Proof. We concentrate on the amount of time required to construct $C(S_i)$ from $C(S_{i-1})$. The first action taken is the determination of p^*, i.e. a weighted nearest point of p_i. This costs $O(n)$ time, as does the determination of an edge and incident face which intersect $pl(p_i, p^*)$. Then $poly(p_i)$ is constructed taking $O(i + h_i)$ time, where h_i denotes the number of edges of $C(S_{i-1})$ contained in the interior of $poly(p_i)$. Deleting these h_i edges and the $O(h_i)$ faces and vertices which are also contained in the interior of $poly(p_i)$ costs $O(h_i)$ time. Thus, for successively inserting the weighted points p_i, for $i = 1, \ldots, n$, the algorithm takes

$O(n^2) + O(h_1 + \ldots + h_n)$ time. The latter sum is in $O(n^2)$ as each face, edge or vertex deleted has to be constructed first. This completes the argument.

Lemma 4–2 implies the main result of this paper.

Theorem 4–3. Let S denote a set of n weighted points in the plane. Then there exists an algorithm which constructs the WVD(S) in $O(n^2)$ time.

Proof. The WVD(S) is computed by (1) embedding S in the XY-plane, (2) constructing the cell complex $C(S)$, (3) intersecting $C(S)$ with the sphere whose inverted image is the XY-plane, and finally (4) inverting the intersection which yields the WVD(S) in the XY-plane. Step (1) requires $O(n)$ time, Step (2) needs $O(n^2)$ time, due to Lemma 4–2, and Steps (3) and (4) can obviously be carried out in time proportional to the size of $C(S)$. The asserted bound follows which completes the argument.

Due to Lemma 2–4, the presented algorithm is asymptotically optimal in the worst case.

5. DISCUSSION

Let us first review the main contributions of this paper. Most important, an optimal algorithm is outlined which constructs the weighted Voronoi diagram of a finite set of points in the plane. To this end, a geometric transform which embeds the diagram in three dimensions and then inverts it is employed. We believe that the optimal algorithm for computing the weighted Voronoi diagram is of value to several areas of science outside computer science (see Boots[13]).

Clearly, the technique presented for computing the planar diagram is not optimal if applied to the one-dimensional case. This shortcoming is not considered important as an optimal algorithm is described in Aurenhammer[19] which constructs the weighted Voronoi diagram of a set of n weighted points on a line in $O(n \log n)$ time. Nevertheless, we conjecture that our technique generalizes nicely to three and higher dimensions, where no results are known yet.

In some situations one might wish to preprocess a set S of n weighted planar points such that for a given query point x one can easily determine those points in S which minimize the weighted distance to x. This can be done by constructing the weighted Voronoi diagram and then superimposing upon it Preparata's[20] structure for point location search. The region (or edge or vertex) x is in (or on) uniquely defines the weighted nearest point(s) in S. This yields Theorem 5–1.

Theorem 5–1. Let S denote a set of n weighted points in the Euclidean plane. Then there exists a data structure which requires $O(n^2 \log n)$ space and time for construction such that the weighted nearest points in S to a query point can be determined in $O(\log n)$ time.

We raise the question whether this search problem can be solved with $O(n^2)$ space and $O(\log n)$ time for answering a query. In addition, exact bounds on the number of faces, edges and vertices that a weighted Voronoi diagram of n planar points can consist of is of vital interest. Methods similar to those used in Seidel[21] for analyzing the complexity of three- and higher-dimensional Voronoi diagrams might be useful for answering this question.

6. SUMMARY

For a set S of n points in the Euclidean plane that are individually weighted by a positive real constant, the weighted Voronoi diagram of S is considered. It associates a region R to each point p with weight $w(p)$ such that the weighted distance $d_e(x, p)/w(p)$ between the points x in R and p is minimal among the points in S.

The regions are bounded by circular edges and define a subdivision of the plane. In general, they are not simply connected and even non-connected. There are $O(n^2)$ components in the worst case.

By means of the geometric transform called inversion, the planar diagram is embedded in three dimensions. The resulting cell complex turns out to be easier constructed than the original structure. Incrementally inserting the weighted points yields total space and time requirements in $O(n^2)$, which is optimal within a constant factor. From the cell complex, the weighted Voronoi diagram can be derived in time proportional to its size.

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